

Packing dimensions of the divergence points of self-similar measures with the open set condition

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Abstract. Let μ be the self-similar measure supported on the self-similar set K with open set condition. In this article, we discuss the packing dimension of the set $\{x \in K : A(\frac{\log \mu(B(x,r))}{\log r}) = I\}$ for $I \subseteq \mathbb{R}$, where $A(\frac{\log \mu(B(x,r))}{\log r})$ denotes the set of accumulation points of $\frac{\log \mu(B(x,r))}{\log r}$ as $r \searrow 0$. Our main result solves the conjecture about packing dimension posed by Olsen and Winter [11] and generalizes the result in [2].

Keywords and phrases Self-similar measures; OSC; Moran structure.

1 Introduction and statement of results

Let $S_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ for $i = 1, \dots, N$ be contracting similarities with contraction ratios $r_i \in (0, 1)$ and let (p_1, \dots, p_N) be a probability vector. Let K and μ be the self-similar set and the self-similar measure associated with the list $(S_1, \dots, S_N, p_1, \dots, p_N)$, namely, K is the unique non-empty compact subset of \mathbb{R}^d such that

$$K = \bigcup_i S_i(K)$$

and μ is the unique Borel probability measure on \mathbb{R}^d such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}. \quad (1.1)$$

As we all known that $\text{supp} \mu = K$. The Open Set Condition (OSC) is fulfilled means that if there exists an open non-empty and bounded subset U of \mathbb{R}^d such that $\bigcup_i S_i U \subseteq U$ and $S_i U \cap S_j U = \emptyset$ for all i, j with $i \neq j$.

During the past 20 years the multifractal structure of μ has attracted considerable attention. Multifractal analysis refers to the study of the fractal geometry of the set $\left\{x \in \mathbb{R}^d : \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha\right\}$. Define the Hausdorff multifractal spectrum $f_H(\alpha)$ of μ and the packing multifractal spectrum $f_P(\alpha)$ of μ as below. For $\alpha \geq 0$, put

$$f_H(\alpha) = \dim_H \left\{x \in \mathbb{R}^d : \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha\right\}, \quad (1.2)$$

and

$$f_P(\alpha) = \dim_P \left\{x \in \mathbb{R}^d : \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha\right\}. \quad (1.3)$$

Define the function $\beta(q) : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\sum_{i=1}^N p_i^q r_i^{\beta(q)} = 1. \quad (1.4)$$

The Legendre transformation of a real valued function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\varphi^*(x) = \inf_y (xy + \varphi(y))$. Arbeiter and Patzschke [1][12] succeed in computing the multifractal spectra $f_H(\alpha)$ and $f_P(\alpha)$ under the OSC.

Theorem 1.1. (see [1] [12]) *Assume that the OSC is satisfied. Then the multifractal spectra $f_H(\alpha)$ and $f_P(\alpha)$ are given by*

$$f_H(\alpha) = f_P(\alpha) = \beta^*(\alpha), \quad \text{for } \alpha \geq 0.$$

Remark 1.1. *Let $s = \dim_H K = \dim_P K$.*

- (1) $\beta(q)$ is a strong decreasing function and $\beta(1) = 0, \beta(0) = s$.
- (2) It is either $\beta(q) = s(1 - q)$ for $p_i = r_i^s, i = 1, 2, \dots, N$ or $\beta(q)$ is convex.
- (3) α can considered as a function about q , and $\alpha = \alpha(q)$ is either $\forall q, \alpha(q) = s$ or a strong decreasing function.

However, the limit $\lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}$ may not exist. Points x for which the limits do not exist are called divergence points. Strichartz [15] showed that

$$\mu \left\{ x \in K : \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} < \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \right\} = 0.$$

However, Barreira and Schmeling [3] and Chen and Xiong [5] proved that

$$\left\{ x \in K : \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} < \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \right\}$$

has full Hausdorff dimension for $(p_1, \dots, p_N) \neq (r_1^s, \dots, r_N^s)$, where $s = \dim_H K$. Chen and Xiong obtained the above result under the strong separation condition. Xiao, Wu and Gao [16] proved that Chen and Xiong's result remains valid under the OSC. These results shows that the set of divergence points has an extremely rich and intricate fractal structure. Hence, Dividing divergence points into different level sets like multifractal analysis has attracted an enormous interest in the mathematical literature.

For $x \in K$, set

$$A(D(x)) = \{y \in (0, +\infty) : \lim_{k \rightarrow \infty} D_{r_k}(x) = y \text{ for some } \{r_k\}_k \searrow 0\},$$

where $D_r(x) = \frac{\log \mu(B(x, r))}{\log r}$.

We now state Olsen and Winter's results and I. Baek, L. Olsen and N. Snigireva's result as follows.

Theorem 1.2. [11] *Assume that the strong separation condition is satisfied. Let $\alpha_{\min} = \min_i \frac{\log p_i}{\log r_i}$ and $\alpha_{\max} = \max_i \frac{\log p_i}{\log r_i}$.*

(1) *If $I \subseteq \mathbb{R}$ is not a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\{x \in K : A(D(x)) = I\} = \emptyset.$$

(2) *If $I \subseteq \mathbb{R}$ is a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\dim_H \{x \in K : A(D(x)) = I\} = \inf_{\alpha \in I} \beta^*(\alpha).$$

Theorem 1.3. [11] *Assume that the strong separation condition is satisfied. If $C \subset \mathbb{R}$ is an arbitrary subset of \mathbb{R} , then*

$$\dim_H \{x \in K : A(D(x)) \subseteq I\} = \dim_P \{x \in K : A(D(x)) \subseteq I\} = \sup_{\alpha \in C} \beta^*(\alpha).$$

Theorem 1.4. [2] *Assume that the strong separation condition is satisfied.*

(1) *If $I \subseteq \mathbb{R}$ is not a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\{x \in K : A(D(x)) = I\} = \emptyset.$$

(2) *If $I \subseteq \mathbb{R}$ is a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\dim_P\{x \in K : A(D(x)) = I\} = \sup_{\alpha \in I} \beta^*(\alpha).$$

Olsen and Winter [11] conjectured that the results in Theorem 1.2 and Theorem 1.3 remain valid even if the strong separation condition is replaced by the OSC. J. Li, M. Wu and Y. Xiong [10] proved that Theorem 1.2 remains valid even if the strong separation condition is replaced by OSC. This article solves Olsen and Winter's conjecture about packing dimension positively. Namely, we will prove that Theorem 1.3 and Theorem 1.4 remain valid even if the strong separation condition is replaced by OSC. More precisely, this paper has the following main results.

Theorem 1.5. *Assume that the OSC is satisfied.*

(1) *If $I \subseteq \mathbb{R}$ is not a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\{x \in K : A(D(x)) = I\} = \emptyset.$$

(2) *If $I \subseteq \mathbb{R}$ is a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\dim_P\{x \in K : A(D(x)) = I\} = \dim_P\{x \in K : A(D(x)) \subseteq I\} = \sup_{\alpha \in I} \beta^*(\alpha).$$

Corollary 1.1. *Assume that the OSC is satisfied. If $I \subseteq \mathbb{R}$ is a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$, then*

$$\dim_H\{x \in K : A(D(x)) \subseteq I\} = \sup_{\alpha \in I} \beta^*(\alpha).$$

Remark 1.2. *Here, singleton is regarded as a special closed subinterval.*

2 Preliminaries

We begin by introducing the definition of packing dimension, which is referred to [6]. Let $0 \leq s < \infty$. For $A \subseteq \mathbb{R}^d$ and $0 < \delta < \infty$, put $P_\delta^s(A) = \sup \sum_i d(B_i)^s$, where $d(B_i)$ denotes the diameter of B_i and the supremum is taken over all disjoint families of closed balls $\{B_1, B_2, \dots\}$ such that $d(B_i) \leq \delta$ and the centres of the B_i 's are in A . Define

$$P^s(A) = \lim_{\delta \rightarrow 0} P_\delta^s(A),$$

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{i=1}^{\infty} P^s(A_i) : A = \bigcup_{i=1}^{\infty} A_i \right\},$$

and

$$\begin{aligned}\dim_P A &= \inf\{s : \mathcal{P}^s(A) = 0\} = \inf\{s : \mathcal{P}^s(A) < \infty\} \\ &= \sup\{s : \mathcal{P}^s(A) > 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\}.\end{aligned}$$

Fix $N \in \mathbb{N}$, for $n = 1, 2, \dots$, set $\Sigma^n = \{1, \dots, N\}^n$ that is, Σ^n is the family of all lists $u = u_1 \cdots u_n$ of length n with entries $u_j \in \{1, \dots, N\}$ and let $\Sigma^* = \bigcup_n \Sigma^n$. For $u \in \Sigma^n$, write $|u| = n$ for the length of u . For $u \in \Sigma^*$, u^- is the word obtained from u by dropping the last letter. Finally, if $u = u_1 \cdots u_n \in \Sigma^n$, write $S_u = S_{u_1} \circ \cdots \circ S_{u_n}$, $p_u = p_{u_1} \cdots p_{u_n}$ and $r_u = r_{u_1} \cdots r_{u_n}$, put $K_u = S_u K$ and set $r_{\min} = \min_{1 \leq i \leq N} r_i$.

Let μ be a Borel probability measure on \mathbb{R}^d admitting compact support. For any open set $V \subseteq \mathbb{R}^d$ with $\mu(V) > 0$, put

$$\Theta_V(q; r) = \sup_i \mu(B(x_i, r))^q, \quad r > 0, q \in \mathbb{R},$$

where the supremum is taken over all families of disjoint closed balls $\{B(x_i, r)\}_i$ contained in V with $x_i \in \text{supp} \mu$. The L^q -spectrum $\tau_V(q)$ of μ on V is defined by

$$\tau_V(q) = \liminf_{r \rightarrow 0} \frac{\log \Theta_V(q; r)}{-\log r}. \quad (2.5)$$

Particularly, for $V = \mathbb{R}^d$, simplify $\Theta_{\mathbb{R}^d}(q; r)$ as $\Theta(q; r)$, $\tau_{\mathbb{R}^d}(q)$ as $\tau(q)$. Peres and Solomyak [13] showed that for any self-similar measure μ , the limit $\lim_{r \rightarrow 0} \frac{\log \Theta(q; r)}{\log r}$ exists for $q \geq 0$. In particular, Lau [9] obtained that if the OSC is satisfied, then $\tau(q) = \beta(q)$ for any $q > 0$, where $\beta(q)$ is defined as in (1.4). Riedi [14] proved that (1.4) remains valid for $q \in \mathbb{R}$.

If $(S_i)_{i=1}^N$ satisfies the OSC, then there exists an open, bounded and non-empty set U such that $\bigcup_i S_i(U) \subset U$, $U \cap K \neq \emptyset$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$. It is simple to check that $S_u K \cap S_{u'} U = \emptyset$, for any $u \neq u' \in \Sigma^n$. Therefore, one can find an open ball $U_0 = U(x_0, r_0) \subset U$ with $x_0 \in K$, where $U(x_0, r_0)$ is the open ball of radius r_0 centered at x_0 . And we fix U_0 in this article.

The following two lemmas can be found in [10].

Lemma 2.1. [10] *If μ is a self-similar measure supported on K and satisfies the OSC, then $\tau_{U_0}(q) = \tau(q) = \beta(q)$ for any $q \in \mathbb{R}$.*

Lemma 2.2. [7][10] *If we choose any $q \in \mathbb{R}$ and set $\alpha = -\beta'(q)$, then for any $\delta, \eta > 0$, there exist $d \in (0, \eta)$, $k \geq d^{-\beta^*(\alpha) + \delta(|q|+1)}$ and $u_1, \dots, u_k \in \Sigma^*$ such that*

- (a) $d^{1+\delta} \leq r_{u_i} \leq d^{1-\delta}$ for all $1 \leq i \leq k$.
- (b) $S_{u_i}(U(x_0, 4r_0))$ are disjoint subsets of U_0 .
- (c) $d^{\alpha+3\delta} \leq p_{u_i} \leq d^{\alpha-3\delta}$ for all $1 \leq i \leq k$.

To get the lower bound estimate of the packing dimension in Theorem 1.5, we need to construct a Moran set. Let's present the definition of the Moran set and some results of it. Fix a closed ball $B \subset \mathbb{R}^d$. Let $\{n_k\}_{k \geq 1}$ be a sequence of positive integers. Let $D = \bigcup_{k \geq 0} D_k$ with $D_0 = \{\emptyset\}$ and $D_k = \{\omega = (j_1 j_2 \cdots j_k) : 1 \leq j_i \leq n_i, 1 \leq i \leq k\}$. Suppose that $\mathcal{G} = \{B_\omega : \omega \in D\}$ is a collection of closed balls of radius r_ω in \mathbb{R}^d . We say that \mathcal{G} fulfills the Moran structure provided it satisfies the following conditions:

- $B_\emptyset = B, B_{\omega j} \subset B_\omega$ for any $\omega \in D_{k-1}, 1 \leq j \leq n_k$;
- $B_\omega \cap B_{\omega'} = \emptyset$ for $\omega, \omega' \in D_k$ with $\omega \neq \omega'$;
- $\lim_{k \rightarrow \infty} \max_{\omega \in D_k} r_\omega = 0$;
- For all $\omega\eta \neq \omega'\eta, \omega, \omega' \in D_m, \omega\eta, \omega'\eta \in D_n, m < n$,

$$\frac{r_{\omega\eta}}{r_\omega} = \frac{r_{\omega'\eta}}{r_{\omega'}}.$$

If \mathcal{G} fulfills the above Moran structure, we call

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\omega \in D_n} B_\omega$$

the Moran set associated with \mathcal{G} .

For $k \in \mathbb{N}$, let

$$c_k = \min_{(i_1 \cdots i_k) \in D_k} \frac{r_{i_1 \cdots i_k}}{r_{i_1 \cdots i_{k-1}}}, \quad M_k = \max_{(i_1 \cdots i_k) \in D_k} r_{i_1 \cdots i_k}$$

Lemma 2.3. [8] *For the Moran set F defined as above, suppose furthermore*

$$\lim_{k \rightarrow \infty} \frac{\log c_k}{\log M_k} = 0. \tag{2.6}$$

Then we have

$$\dim_P F = \limsup_{k \rightarrow \infty} s_k,$$

where s_k satisfies the equation $\sum_{\omega \in D_k} r_\omega^{s_k} = 1$.

3 Proof of Theorem 1.5

To (1) in Theorem 1.5, the reader is referred to [10] for a more detailed discussion. Let $K^I = \{x \in K : A(D(x)) \subseteq I\}$, and write $K_I = \{x \in K : A(D(x)) = I\}$. We prove (2) by showing that

$$\dim_P K_I \geq \sup_{\alpha \in I} \beta^*(\alpha), \quad (3.7)$$

and

$$\dim_P K^I \leq \sup_{\alpha \in I} \beta^*(\alpha). \quad (3.8)$$

3.1 Proof of inequality (3.7)

Since I is closed and $\beta^*(\alpha)$ is continuous, there exists $\alpha_0 \in I$ such that $\sup_{\alpha \in I} \beta^*(\alpha) = \beta^*(\alpha_0)$. The idea behind the proof is to construct a Moran set F such that

$$F \subseteq K_I, \quad (3.9)$$

and

$$\dim_P F \geq \beta^*(\alpha_0). \quad (3.10)$$

This approach was used in [2], [4], [7], [8], [10], [11] and the following proof also benefits from these papers.

Let $i \in \mathbb{N}$. Since I is connected, we may choose $q_{i,1}, \dots, q_{i,M_i} \in \mathbb{R}$ such that

- $\alpha_{i,j} \in I$, where $\alpha_{i,j} = -\beta'(q_{i,j})$;
- $I \subseteq \bigcup_{j=1}^{M_i} B(\alpha_{i,j}, \frac{1}{i})$;
- $|\alpha_{i,j} - \alpha_{i,j+1}| \leq \frac{1}{i}$ for all j , $|\alpha_{i,M_i} - \alpha_{i+1,1}| \leq \frac{1}{i}$;
- $\alpha_{i,M_i} = \alpha_0$ for all i .

Remark 3.1. *In fact, $\overline{\{\alpha_{i,j}, \alpha_{i,j+1}, \dots, \alpha_{i,M_i}, \alpha_{i+1,1}, \alpha_{i+1,2}, \dots\}} = I$, for any $i \in \mathbb{N}, 1 \leq j \leq M_i$.*

Choose a positive sequence $(\delta_i)_{i=1}^\infty \searrow 0$. Note that $\tau_{U_0} = \beta(q)$ for any $q \in \mathbb{R}$. It follows from Lemma 2.2 that there exist positive real numbers $(d_{i,j})_{i \in \mathbb{N}, j=1, \dots, M_i}$, $(k_{i,j})_{i \in \mathbb{N}, j=1, \dots, M_i}$ and $\mathcal{B}_{i,j} = \{u_{i,j,s} : 1 \leq s \leq k_{i,j}\} \subset \Sigma^*$ such that

$$(i) \ 1 > d_{1,1} > d_{1,2} > \cdots > d_{1,M_1} > d_{2,1} > d_{2,2} > \cdots > 0.$$

$$(ii) \ k_{i,j} \geq (d_{i,j})^{-\beta^*(\alpha_{i,j})+\delta_i(|q_{i,j}|+1)}.$$

$$(iii) \ (d_{i,j})^{1+\delta_i} \leq r_{u_{i,j},s} \leq (d_{i,j})^{1-\delta_i} \text{ for } 1 \leq s \leq k_{i,j}.$$

$$(iv) \ S_{u_{i,j},s}(U(x_0, 4r_0)) (1 \leq s \leq k_{i,j}) \text{ are disjoint subsets of } U_0.$$

$$(v) \ (d_{i,j})^{\alpha_{i,j}+3\delta_i} \leq p_{u_{i,j},s} \leq (d_{i,j})^{\alpha_{i,j}-3\delta_i} \text{ for } 1 \leq s \leq k_{i,j}.$$

Choose a sequence of positive integers $(N_{k,j})_{k \in \mathbb{N}, j=1, \dots, M_k}$ large enough such that

$$(vi) \ (d_{k,j})^{N_{k,j}} \leq (d_{k,j+1})^{2^k} \text{ for any } k \in \mathbb{N} \text{ and } 1 \leq j < M_k.$$

$$(vii) \ \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^{k-1} \sum_{j=1}^{M_i} N_{i,j} \log d_{i,j} + \sum_{j=1}^{i_k-1} N_{k,j} \log d_{k,j}}{N_{k,i_k} \log d_{k,i_k}} = 0 \text{ for any } 1 \leq i_k \leq M_k.$$

Define a sequence of subsets of Σ^* as follows:

$$\underbrace{\mathcal{B}_{1,1}, \dots, \mathcal{B}_{1,1}}_{N_{1,1} \text{ times}}, \underbrace{\mathcal{B}_{1,2}, \dots, \mathcal{B}_{1,2}}_{N_{1,2} \text{ times}}, \dots, \underbrace{\mathcal{B}_{1,M_1}, \dots, \mathcal{B}_{1,M_1}}_{N_{1,M_1} \text{ times}}, \underbrace{\mathcal{B}_{2,1}, \dots, \mathcal{B}_{2,1}}_{N_{2,1} \text{ times}}, \dots$$

and relabel them as $\{\mathcal{B}_n^*\}_{n=1}^\infty$. Put

$$\mathcal{G} = \{S_{v_1 \dots v_k}(\overline{U_0}) : k \in \mathbb{N}, v_i \in \mathcal{B}_i^* \text{ for } 1 \leq i \leq k\},$$

and set

$$F = \bigcap_{n=1}^\infty \bigcup_{v_1 \in \mathcal{B}_1^*, \dots, v_n \in \mathcal{B}_n^*} S_{v_1 \dots v_n}(\overline{U_0}).$$

It is easy to check that F is a Moran set associated with \mathcal{G} , and F satisfies (3.9). The reader is referred to [10] for a more detailed discussion.

For large n , write $n = \sum_{j=1}^{M_1} N_{1,j} + \cdots + \sum_{j=1}^{i_k} N_{k,j} + p$ with $1 \leq p < N_{k,i_k+1}$. Put

$$A_k = \left(\prod_{i=1}^{k-1} \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(-\beta^*(\alpha_{i,j})+\delta_i(|q_{i,j}|+1))} \right) \prod_{j=1}^{i_k} (d_{k,j})^{N_{k,j}(-\beta^*(\alpha_{k,j})+\delta_k(|q_{k,j}|+1))}.$$

Combining (ii) and (iii), we have

$$\prod_{s=1}^n \#\mathcal{B}_s^* \geq A_k (d_{k,i_k+1})^{p(-\beta^*(\alpha_{k,i_k+1})+\delta_k(|q_{k,i_k+1}|+1))} \quad (3.11)$$

and

$$\inf_{v_n \in \mathcal{B}_n^*} r_{v_n} \geq (d_{k,i_k+1})^{1+\delta_k},$$

$$\sup_{v_1 \in \mathcal{B}_1^*, \dots, v_n \in \mathcal{B}_n^*} r_{v_1 \dots v_n} \leq \left(\prod_{i=1}^{k-1} \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(1-\delta_i)} \right) \prod_{j=1}^{i_k} (d_{k,j})^{N_{k,j}(1-\delta_k)} (d_{k,i_k+1})^{p(1-\delta_k)}. \quad (3.12)$$

Using (3.12) and (vi), we have

$$\lim_{n \rightarrow \infty} \frac{\log \left(\inf_{v_n \in \mathcal{B}_n^*} r_{v_n} \right)}{\log \left(\sup_{v_1 \in \mathcal{B}_1^*, \dots, v_n \in \mathcal{B}_n^*} r_{v_1 \dots v_n} \right)} = 0.$$

This implies that the condition (2.6) in Lemma 2.3 is fulfilled. Hence, by Lemma 2.3, we conclude that $\dim_P F = \limsup_{n \rightarrow \infty} s_n$, where

$$\sum_{v_1 \in \mathcal{B}_1^*, \dots, v_n \in \mathcal{B}_n^*} (r_{v_1 \dots v_n})^{s_n} = 1.$$

It follows that

$$\dim_P F \geq \limsup_{n \rightarrow \infty} \frac{\log \left(\prod_{s=1}^n \# \mathcal{B}_s^* \right)}{-\log \left(\inf_{v_1 \in \mathcal{B}_1^*, \dots, v_n \in \mathcal{B}_n^*} r_{v_1 \dots v_n} \right)}. \quad (3.13)$$

Next, let's prove (3.10). Choose a special sequence of positive integers $\{n_t\}_{t \geq 1}$, with $n_t = \sum_{j=1}^{M_1} N_{1,j} + \dots + \sum_{j=1}^{M_t} N_{t,j}$. Combining (ii) and (iii), we get

$$\begin{aligned} \prod_{s=1}^{n_t} \# \mathcal{B}_s^* &= k_{1,1}^{N_{1,1}} k_{1,2}^{N_{1,2}} \dots k_{1,M_1}^{N_{1,M_1}} k_{2,1}^{N_{2,1}} \dots k_{t,M_t}^{N_{t,M_t}} \\ &\geq \prod_{i=1}^t \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(-\beta^*(\alpha_{i,j}) + \delta_i(|q_{i,j}|+1))}. \end{aligned} \quad (3.14)$$

This (3.14), together with (3.13) and the following inequality

$$\inf_{v_1 \in \mathcal{B}_1^*, \dots, v_{n_t} \in \mathcal{B}_{n_t}^*} r_{v_1 \dots v_{n_t}} \geq \prod_{i=1}^t \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(1+\delta_i)},$$

yields

$$\begin{aligned}
\dim_P F &\geq \limsup_{t \rightarrow \infty} \frac{\log \prod_{i=1}^t \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(-\beta^*(\alpha_{i,j}) + \delta_i(|q_{i,j}|+1))}}{-\log \prod_{i=1}^t \prod_{j=1}^{M_i} (d_{i,j})^{N_{i,j}(1+\delta_i)}} \\
&= \limsup_{t \rightarrow \infty} \frac{\sum_{i=1}^t \sum_{j=1}^{M_i} N_{i,j} (-\beta^*(\alpha_{i,j}) + \delta_i(|q_{i,j}|+1)) \log d_{i,j}}{-\sum_{i=1}^t \sum_{j=1}^{M_i} N_{i,j} (1 + \delta_i) \log d_{i,j}} \\
&\geq \beta^*(\alpha_0).
\end{aligned}$$

3.2 Proof of inequality (3.8)

The idea of the following proof comes from an article by Patzschke [12]. First, present a lemma as below. Put $\Xi_{\leq \alpha} = \left\{ x \in K : \limsup_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} \leq \alpha \right\}$ and write $\Xi_{\geq \alpha} = \left\{ x \in K : \liminf_{r \rightarrow 0} \frac{\log \mu(B(x,r))}{\log r} \geq \alpha \right\}$.

Lemma 3.1. *For $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,*

- (i) *if $\alpha \leq \alpha(0)$, then $\dim_P \Xi_{\leq \alpha} \leq \beta^*(\alpha)$.*
- (ii) *if $\alpha \geq \alpha(0)$, then $\dim_P \Xi_{\geq \alpha} \leq \beta^*(\alpha)$.*

Proof. (i) If $\alpha \leq \alpha(0)$, then $q \geq 0$. Write $\beta = \beta(q)$. Let $\epsilon > 0$ and $0 < \rho < \frac{1}{2}$ and define

$$\Xi_{< \alpha, m} = \{x \in K : \rho^{n(\alpha+\epsilon)} \leq \mu(B(x, \rho^n)) \text{ for all } n \geq m\}$$

for $m \in \mathbb{N}$. Then $\Xi_{\leq \alpha} \subseteq \bigcup_{m=1}^{\infty} \Xi_{< \alpha, m}$.

Let $B(x_1, r_1^*), B(x_2, r_2^*), \dots$ be a ρ^m -packing of $\Xi_{< \alpha, m}$. For $i \in \mathbb{N}$ choose $n_i \in \mathbb{N}$ such that $\rho^{n_i} \leq r_i^* < \rho^{n_i-1}$. Then $n_i \geq m$ and $B(x_i, \rho^{n_i}) \subseteq B(x_i, r_i^*) \subseteq B(x_i, \rho^{n_i-1})$. Hence, the sequence $B(x_1, \rho^{n_1}), B(x_2, \rho^{n_2}), \dots$ consists of disjoint sets. For $0 < r < 1$, define $\Gamma_r = \{u \in \Sigma^* : r_u < r \leq r_{u-}\}$. It is well known that the OSC implies that $\sup_{x \in \mathbb{R}^d, 0 < r < 1} \#\{u \in \Gamma_r : K_u \cap B(x, r) \neq \emptyset\} < \infty$. Then $\mu(B(x, r)) \leq c_1 \max\{p_u : u \in \Gamma_r, K_u \cap B(x, r) \neq \emptyset\}$ for all $n \in \mathbb{N}$ and all $x \in K$. For $n \in \mathbb{N}$ write $\Gamma(n) = \Gamma_{\rho^n}$. By volume estimating we obtain a constant c_2 such that

$$\#\{i = 1, 2, \dots : n_i = n, B(x_i, \rho^{n_i}) \cap K_u \neq \emptyset\} \leq c_2$$

for all $u \in \Gamma(n)$ and all n . Therefore, using the definition of $\Xi_{< \alpha, m}$,

$$\begin{aligned}
& \sum_{i=1}^{\infty} d(B(x_i, r_i^*))^{\alpha q + \beta + \epsilon(1+q)} \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1+q)} \sum_{i=1}^{\infty} \rho^{n_i(\beta + \epsilon)} \rho^{n_i q(\alpha + \epsilon)} \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1+q)} \sum_{i=1}^{\infty} \rho^{n_i(\beta + \epsilon)} \mu(B(x_i, \rho^{n_i}))^q \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1+q)} c_1^q c_2 \sum_{n=m}^{\infty} \sum_{u \in \Gamma(n)} \rho^{n(\beta + \epsilon)} p_u^q \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1+q)} c_1^q c_2 r_{\min}^{-1(\beta + \epsilon)} \sum_{k=1}^{\infty} \left(\sum_{i=1}^N p_i^q r_i^{\beta(q) + \epsilon} \right)^k
\end{aligned}$$

This shows, that $\dim_P \Xi_{< \alpha, m} \leq \alpha q + \beta(q) + \epsilon(1+q)$ for all $m \in \mathbb{N}$ and hence, $\dim_P \Xi_{\leq \alpha} \leq \alpha q + \beta(q) + \epsilon(1+q)$. Since $\epsilon > 0$ was arbitrary, $\dim_P \Xi_{\leq \alpha} \leq \inf\{\alpha q + \beta(q)\}$.

(ii) Let $\alpha \geq \alpha(0)$. Let $q < 0$ and write $\beta = \beta(q)$. Let $\epsilon > 0$ and $0 < \rho < \frac{1}{2}$ and define

$$\Xi_{> \alpha, m} = \{x \in K : \rho^{n(\alpha - \epsilon)} \geq \mu(B(x, \rho^n)) \text{ for all } n \geq m\}$$

for $m \in \mathbb{N}$. Then $\Xi_{\geq \alpha} \subseteq \bigcup_{m=1}^{\infty} \Xi_{> \alpha, m}$. Let $B(x_1, r_1^*), B(x_2, r_2^*), \dots$ be a ρ^m -packing of $\Xi_{> \alpha, m}$. For $i \in \mathbb{N}$ choose $n_i \in \mathbb{N}$ such that $\rho^{n_i} \leq r_i^* < \rho^{n_i - 1}$. Then $n_i \geq m$ and $B(x_i, \rho^{n_i}) \subseteq B(x_i, r_i^*) \subseteq B(x_i, \rho^{n_i - 1})$. Hence, the sequence $B(x_1, \rho^{n_1}), B(x_2, \rho^{n_2}), \dots$ consists of disjoint sets. Using similar steps as above, then

$$\begin{aligned}
& \sum_{i=1}^{\infty} d(B(x_i, r_i^*))^{\alpha q + \beta + \epsilon(1-q)} \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1-q)} \sum_{n=m}^{\infty} \sum_{u \in \Gamma(n)} \rho^{n(\beta + \epsilon)} p_u^q \\
& \leq \left(\frac{2}{\rho}\right)^{\alpha q + \beta + \epsilon(1-q)} r_{\min}^{-1(\beta + \epsilon)} \sum_{k=1}^{\infty} \left(\sum_{i=1}^N p_i^q r_i^{\beta(q) + \epsilon} \right)^k.
\end{aligned}$$

Finally, it follows that $\dim_P \Xi_{\geq \alpha} \leq \beta^*(\alpha)$. This completes the proof of lemma.

If $\sup I \leq \alpha(0)$, then $K^I \subseteq \Xi_{\leq \sup I}$. So, $\dim_P K^I \leq \sup_{\alpha \in I} \beta^*(\alpha)$.

If $\inf I \geq \alpha(0)$, then $K^I \subseteq \Xi_{\geq \inf I}$. So, $\dim_P K^I \leq \sup_{\alpha \in I} \beta^*(\alpha)$.

If $\alpha(0) \in I$, then $\dim_P K^I = \dim_H K^I = \beta^*(\alpha(0))$.

3.3 Proof of corollary 1.1

It follows from the following facts:

- For any $\alpha \in I$, we have that $\{x \in K : A(D(x)) = \alpha\} \subseteq K^I$.
- $\dim_H K^I \leq \dim_P K^I$.

Remark 3.2. For K^I , " I " can be any subset of \mathbb{R} . In fact, $K^I = K^{I \cap [\alpha_{\min}, \alpha_{\max}]}$.

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